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The Pauli equation for probability distributions

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Abstract

The tomographic-probability distribution for a measurable coordinate and spin projection is introduced to describe quantum states as an alternative to the density matrix. An analogue of the Pauli equation for the spin-$\frac{1}{2}$ particle is obtained for such a probability distribution instead of the usual equation for the wavefunction. Examples of the tomographic description of Landau levels and coherent states of a charged particle moving in a constant magnetic field are presented.

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1. Introduction

Since the early days of quantum mechanics, we have been forced to coexist with complex probability amplitudes without worrying about their lack of any reasonable physical meaning. One should not ignore, however, that the wave-like properties of quantum objects still raise conceptual problems, on whose solutions a general consensus is far from having been reached [1, 2].

A possible way out of this difficulty has been implicitly suggested by Feynman [3], who has shown that, by dropping the assumption that the probability for an event must always be non-negative, one can avoid the use of probability amplitudes in quantum mechanics. This proposal goes back to the work of Wigner [4], who first introduced non-positive pseudoprobabilities to represent quantum mechanics in phase space, and to the Moyal approach to quantum mechanics [5].

From a conceptual point of view, the elimination of the waves from quantum theory is in line with the procedure inaugurated by Einstein with the elimination of the ether in the theory of electromagnetism.

The phase-space formulation of quantum mechanics [4, 6, 7] provides a means of analysing quantum-mechanical systems while still employing a classical framework. Moreover, a quantum mechanics without wavefunctions has been discussed in several papers [8].

Recently, the problem of quantum state measurement, initially posed by Pauli [9], received a lot of attention [10]. The tomographic approach [11, 12] to the quantum state of a system has allowed one to establish a map between the density operator (or any its representation) and a set of probability distributions, often called 'marginals'. The latter have all the characteristics of classical probabilities; they are non-negative, measurable and normalized.

Based on this connection, a classical-like description of quantum dynamics by means of 'symplectic tomography' has been formulated [13], providing a bridge between classical and quantum worlds. That is, the evolution of a quantum system with continuous observables (namely, quadrature components of a field mode) was described in terms of a classical-like equation for a marginal distribution. Different aspects of this classical-like description using tomographic probabilities were recently analysed [14, 15].

On the other hand, discrete observables, such as spin or angular momentum, are as important in quantum mechanics as the continuous ones. Hence, the tomography scheme for discrete variables was introduced [16], and the marginal distribution for rotated spin variables has been constructed [17], deriving an evolution equation for this function. The same avenue has been followed recently in [18].

Here, we would extend the approach by considering a spin-$\frac{1}{2}$ particle moving in a potential, then constructing the marginal distributions for space coordinates and spin projections, and finally deriving the evolution equation for such probabilities, which would be an analogue of the Pauli equation. It would also be a generalization of approaches attempted in our previous papers [13].

Essentially, our aim is to eliminate the hybrid procedure of describing the dynamical evolution of a system, which consists of the first stage where the theory provides a deterministic evolution of the wavefunction, followed by a hand-made construction of the physically meaningful probability distributions. If the probabilistic nature of the microscopic phenomena is fundamental and not simply due to our ignorance, as in classical statistical mechanics, why should it be impossible to describe them in probabilistic terms from the very beginning? On the other hand, the language of probability, suitably adapted to take into account all the relevant constraints, seems to be the only language capable of expressing the fundamental role of 'chance' in nature [19].

The paper is organized as follows. In section 2, we review the general approach to construct known tomography schemes using a density matrix in the specifically transformed reference frames. In section 3, we derive the general evolution equation for tomographic probabilities (marginal distributions), which describe the quantum state instead of the density matrix. In section 4, the general scheme of tomography construction is used to rederive the particular example of symplectic tomography, which is applied for measuring states depending on continuous quadrature. In section 5, the general scheme is used to rederive the construction of spin-state tomography. In section 6, the general scheme of section 2 is applied to obtain tomographic probabilities in the combined situation described by spatial (multidimensional as well) and spin variables. In section 7, some examples are studied in the context of the probability representation of quantum mechanics. Section 8 gives conclusions. Herein, we use natural units ($\hbar = c = 1$).

2. General approach to quantum tomography

In this section, we give a short review of the general principles used to construct a tomography scheme for measuring quantum states. Recently, we established a quite general principle of constructing measurable probabilities, which completely determine the quantum state in the tomographic approach [20]; more refined treatments then followed [21,22]. Here, we apply our general approach to derive the evolution equation for the tomographic probabilities, which is an alternative in some sense to the Schrödinger equation for the wavefunction (or the quantum Liouville equation for the density matrix).

Let us consider a quantum state described by the density operator $\hat{\rho}$, which is a non-negative Hermitian operator, i.e.

$$\hat{\rho}^{\dagger} = \hat{\rho} \qquad \operatorname{Tr} \hat{\rho} = 1 \tag{1}$$

and

$$\langle v|\hat{\rho}|v\rangle = \rho_{v,v} \geqslant 0. \tag{2}$$

We label the vector basis $|v\rangle$ in the space of pure quantum states by the multidimensional index $v = (v_1, v_2, \ldots, v_N)$, where the number N shows the number of degrees of freedom of the system under consideration. Among indices v_k, $k = 1, \ldots, N$, there are continuous ones such as position (or momentum) and discrete ones such as spin projections. In this sense, the wavefunction $\psi(v) = \langle v|\psi\rangle$ of a pure state $|\psi\rangle$ depends both on continuous and discrete observables. Formula (2) can be rewritten by using the Hermitian projection operator

$$\hat{\Pi}_v = |v\rangle\langle v| \tag{3}$$

in the following form:

$$\rho_{v,v} = \operatorname{Tr}\left\{\hat{\Pi}_v \hat{\rho}\right\}. \tag{4}$$

The physical meaning of the projector $\hat{\Pi}_v$ is that it extracts the state $|v\rangle$ with given v (for example, with given position and spin projection), which is an eigenstate of the commuting Hermitian operators $\hat{V} = \left(\hat{V}_1, \hat{V}_2, \ldots, \hat{V}_N\right)$

$$\hat{V}_k|v\rangle = v_k|v\rangle. \tag{5}$$

In the space of states, there is a family of unitary transformation operators $\hat{U}(\sigma)$ depending on the parameters $\sigma = (\sigma_1, \ldots, \sigma_k \ldots)$, that can be sometimes identified with group-representation operators. In these cases, the parameters σ describe the group element. It was shown [20, 23] that known tomography schemes can be considered from the viewpoint of group theory by using appropriate groups. More recently this concept has been developed obtaining an elegant group-theoretical approach to quantum-state measurement [22]. Here, we formulate the tomographic approach in the following way. Let us introduce a 'transformed density operator'

$$\hat{\rho}_\sigma = \hat{U}^{-1}(\sigma)\hat{\rho}\hat{U}(\sigma). \tag{6}$$

Its diagonal elements are still non-negative probabilities

$$\langle z|\hat{\rho}_\sigma|z\rangle = \langle\langle z|\hat{\rho}|z\rangle\rangle \equiv w(z, \sigma) \tag{7}$$

where $|z\rangle$ is one of the possible vectors $|v\rangle$, while the symbol $|z\rangle\rangle$ denotes the transformed vectors

$$|z\rangle\rangle = \hat{U}(\sigma)|z\rangle \tag{8}$$

which in turn are eigenstates of the transformed operators

$$\hat{Z} = \hat{U}(\sigma)\hat{V}\hat{U}^{-1}(\sigma). \tag{9}$$

As a consequence of the unit trace of the density operator, we also have the normalization condition

$$\int \mathrm{d}z\, w(z,\sigma) = 1. \tag{10}$$

Of course, in the case of discrete indices, the integral in (10) is replaced by a sum over discrete variables.

Formula (7) can be interpreted as the probability density for the measurement of the observable $\hat{V}$ in an ensemble of transformed reference frames labelled by the index σ, if the state $\hat{\rho}$ is given. Along with this interpretation, one can also consider the transformed projector

$$\hat{\Pi}_z(\sigma) = \hat{U}(\sigma)\hat{\Pi}_z\hat{U}^{-1}(\sigma) = |z\rangle\rangle\langle\langle z|. \tag{11}$$

The explicit expression for the probability $w(z,\sigma)$ takes the form

$$w(z,\sigma) = \mathrm{Tr}\left\{\hat{\rho}\hat{\Pi}_z(\sigma)\right\} = \mathrm{Tr}\left\{\hat{\rho}|z\rangle\rangle\langle\langle z|\right\}. \tag{12}$$

These probability densities are also called 'marginal' distributions as a generalization of the concept introduced by Wigner [4]. The tomography schemes are based on the possibility of finding the inverse of equation (12). If it is possible to solve equation (12) by considering the probability $w(y,\sigma)$ as a known function and the density matrix as an unknown operator, the quantum state can be described by the positive probability instead of the density matrix. This property is the essence of state-reconstruction techniques. In such cases, the inverse of (12) takes the form

$$\hat{\rho} = \int \mathrm{d}z\,\mathrm{d}\sigma\; w(z,\sigma)\hat{K}(z,\sigma). \tag{13}$$

Thus, there exists a family of operators $\hat{K}(z,\sigma)$ depending on both the variables z and the parameters σ such that the density operator is reconstructed, if the probability $w(z,\sigma)$ is known. It is worth remarking that transformations $\hat{U}(\sigma)$ can form other algebraic constructions, which do not have the group structure [23]. The only condition for the existence of a tomography scheme is the possibility of inverting (12). In the cases of optical tomography [12] symplectic tomography [13], and spin tomography [17, 24], the sets of transformations $\hat{U}(\sigma)$ have the structure of corresponding Lie groups (i.e. rotation group $O(2)$, symplectic group $Sp(2R)$ and SU_2 group).

3. The time evolution equation

We are now interested in obtaining the evolution equation for the probability $w\big(z,\sigma,t\big)$, in which t is the time parameter. Using equation (12) one has

$$\partial_t\, w(z,\sigma,t) = \mathrm{Tr}\big\{[\partial_t\,\hat{\rho}(t)]\hat{\Pi}_z(\sigma)\big\}. \tag{14}$$

On the other hand, the density operator satisfies the Liouville–von Neumann equation

$$\partial_t\,\hat{\rho}(t) = \mathrm{i}\big[\hat{\rho}(t),\hat{H}\big] \tag{15}$$

where $\hat{H}$ is the Hamiltonian of the system. By inserting equation (15) in (14), and in view of equation (13), we obtain the evolution equation for the probability w in a closed form

$$\partial_t\, w(z,\sigma,t) = \int \mathrm{d}z'\,\mathrm{d}\sigma'\, w(z',\sigma',t)\,\mathrm{Tr}\left\{\mathrm{i}\big[\hat{K}(z',\sigma'),\hat{H}\big]\hat{\Pi}_z(\sigma)\right\}. \tag{16}$$

Equation (16) represents the classical-like version of the Liouville–von Neumann equation, thus, it would be the analogue of the Pauli equation for a system with space and spin degrees of freedom.

4. Quantum tomography with continuous variables

For a one-dimensional system, we consider an operator $\hat{X}$ as the linear combination of position $\hat{q}$ and momentum $\hat{p}$ [25, 26]

$$\hat{X} = \mu\hat{q} + \nu\hat{p} \tag{17}$$

which depends on real parameters μ, ν; X is a measurable observable due to its Hermiticity. Since the linear canonical transformation (17) belongs to the symplectic group $Sp(2, R)$, the tomography scheme under discussion was called 'symplectic tomography' [26, 27].

The probability (marginal) related to the observable (17) is given by

$$w(x, \mu, \nu) = \langle\langle x|\hat{\rho}|x\rangle\rangle \tag{18}$$

where $\hat{\rho}$ is the system's density operator, while the eigenstates $|x\rangle\rangle$ of the operator (17) can be written as

$$|x\rangle\rangle = \int \mathrm{d}q\ \langle q|x\rangle\rangle\ |q\rangle \tag{19}$$

with $|q\rangle$ the position eigenkets. The wavefunction $\langle q|x\rangle\rangle$ can be easily calculated by using the following equality:

$$\langle q|\hat{X}|x\rangle\rangle = \langle q|\mu\hat{q} + \nu\hat{p}|x\rangle\rangle. \tag{20}$$

Equation (20) can be transformed into the partial differential equation

$$x\ \langle q|x\rangle\rangle = \mu q\ \langle q|x\rangle\rangle - \mathrm{i}\nu\frac{\partial}{\partial q}\langle q|x\rangle\rangle. \tag{21}$$

The solution is

$$\langle q|x\rangle\rangle = \left(\frac{\mathrm{i}}{2\pi\nu}\right)^{1/2} \exp\left[\mathrm{i}\frac{x}{\nu}q - \frac{\mathrm{i}}{2}\frac{\mu}{\nu}q^2\right]. \tag{22}$$

It is worth noting that as soon as $\mu \to 1$ and $\nu \to 0$, the transformed position state $|x\rangle\rangle \to |x\rangle$ and the wavefunction (22) tends to $\delta(q - x)$.

Furthermore, equation (18) can be formally rewritten as

$$w(x, \mu, \nu) = \mathrm{Tr}\left\{\hat{\rho}\hat{\Pi}_x(\mu, \nu)\right\} \tag{23}$$

with the transformed projector being given by

$$\hat{\Pi}_x(\mu, \nu) = \hat{U}(\mu, \nu)\hat{\Pi}_x\hat{U}^{-1}(\mu, \nu) \qquad \hat{\Pi}_x = |x\rangle\langle x| \tag{24}$$

where the transformation $\hat{U}(\sigma)$ is chosen to be the symplectic-group representation [20]

$$\hat{U}(\mu, \nu) = \exp\left[\frac{\mathrm{i}\lambda}{2}\left(\hat{q}\hat{p} + \hat{p}\hat{q}\right)\right]\exp\left[\mathrm{i}\phi\left(\frac{\hat{p}^2}{2} + \frac{\hat{q}^2}{2}\right)\right]. \tag{25}$$

The rotation and scaling parameters ϕ and λ are related to μ and ν by the following formulae:

$$\begin{aligned} &\mu = \mathrm{e}^{\lambda}\cos\phi && \nu = \mathrm{e}^{-\lambda}\sin\phi \\ &\phi = \tfrac{1}{2}\arcsin(2\mu\nu) && \mathrm{e}^{2\lambda} = \frac{1 - \sqrt{1 - 4\mu^2\nu^2}}{2\nu^2}. \end{aligned} \tag{26}$$

This means that the marginal distribution $w(x,\ \mu,\ \nu)$, for this particular case of symplectic tomography, is given by the relationship

$$w(x,\mu,\nu) = \mathrm{Tr}\left\{|x\rangle\langle x|\exp\left[-\mathrm{i}\phi\left(\frac{\hat{p}^2}{2}+\frac{\hat{q}^2}{2}\right)\right]\exp\left[-\frac{\mathrm{i}\lambda}{2}(\hat{q}\hat{p}+\hat{p}\hat{q})\right]\right.$$
$$\left.\times\ \hat{\rho}\exp\left[\frac{\mathrm{i}\lambda}{2}(\hat{q}\hat{p}+\hat{p}\hat{q})\right]\exp\left[\mathrm{i}\phi\left(\frac{\hat{p}^2}{2}+\frac{\hat{q}^2}{2}\right)\right]\right\}. \qquad (27)$$

Such measurable probability can be expressed explicitly as [27]

$$w(x,\mu,\nu) = \int \mathrm{d}y\,\mathrm{d}k\ \exp\left[-\mathrm{i}kx+\frac{\mathrm{i}\mu\nu k^2}{2}+\mathrm{i}ky\mu\right]\rho(y+\nu k,y) \qquad (28)$$

where $\rho(y+\nu k, y) = \langle y+\nu k|\hat{\rho}|y\rangle$ is the representation of the density matrix over the position eigenkets. The marginal satisfies the following homogeneous property:

$$w(x,\mu,\kappa\nu) = \frac{1}{|\kappa|}\,w\left(\frac{x}{\kappa},\frac{\mu}{\kappa},\nu\right) \qquad w(x,\kappa\mu,\nu) = \frac{1}{|\kappa|}\,w\left(\frac{x}{\kappa},\mu,\frac{\nu}{\kappa}\right). \qquad (29)$$

The above relation (28) can be inverted [26] as

$$\hat{\rho} = \int \mathrm{d}x\,\mathrm{d}\mu\,\mathrm{d}\nu\ w(x,\mu,\nu)\,\hat{K}(x,\mu,\nu) \qquad (30)$$

where the kernel operator takes the form

$$\hat{K}(x,\mu,\nu) = \frac{1}{2\pi}\epsilon^2\exp\left[-\mathrm{i}\epsilon X+\frac{\mathrm{i}\epsilon^2\mu\nu}{2}\right]\mathrm{e}^{\mathrm{i}\epsilon\mu\hat{q}}\,\mathrm{e}^{\mathrm{i}\epsilon\nu\hat{p}}. \qquad (31)$$

Here, ϵ can be set equal to unity; this freedom reflects the overcompleteness of the information obtainable by means of all possible marginals (27) [25, 26].

The multimode generalization [26] is straightforward, and the analogue of formula (27) holds with the following replacement:

$$\begin{aligned}
&|x\rangle \longrightarrow |\vec{x}\,\rangle \qquad \vec{x} = (x_1, x_2, \ldots)\\
&\phi\left(\frac{\hat{p}^2}{2}+\frac{\hat{q}^2}{2}\right) \longrightarrow \phi_1\left(\frac{\hat{p}_1^2}{2}+\frac{\hat{q}_1^2}{2}\right)+\phi_2\left(\frac{\hat{p}_2^2}{2}+\frac{\hat{q}_2^2}{2}\right)+\cdots\\
&\lambda(\hat{q}\hat{p}+\hat{p}\hat{q}) \longrightarrow \lambda_1(\hat{q}_1\hat{p}_1+\hat{p}_1\hat{q}_1)+\lambda_2(\hat{q}_2\hat{p}_2+\hat{p}_2\hat{q}_2)+\cdots.
\end{aligned} \qquad (32)$$

Relations of the parameters λ_k and ϕ_k to the parameters μ_k and ν_k are the same as given by equation (26).

5. Quantum tomography with discrete variables

In this section, we consider a spin-j system. Following [17, 24] we derive the expression for the density matrix of a spin state in terms of measurable probability distributions.

For arbitrary values of spin, let the spin state have the density matrix

$$\rho^{(j)}_{mm'} = \langle jm|\hat{\rho}^{(j)}|jm'\rangle \qquad m = -j, -j+1, \ldots, j-1, j \qquad (33)$$

where

$$\hat{j}_3|jm\rangle = m|jm\rangle \qquad \hat{j}^2|jm\rangle = j(j+1)|jm\rangle \qquad (34)$$

and

$$\hat{\rho}^{(j)} = \sum_{m=-j}^{j} \sum_{m'=-j}^{j} \rho_{mm'}^{(j)} |jm\rangle\langle jm'|. \tag{35}$$

The operator $\hat{\rho}^{(j)}$ is the density operator of the state under consideration.

The general group construction of tomographic schemes [20] was also used for spin tomography [17, 24]. The idea is to consider the diagonal elements of the density matrix $\hat{\rho}$ in another reference frame, i.e. in the rotated reference frame. To this end, we introduce a rotated measurable spin projection

$$\hat{J}_3(\alpha, \beta, \gamma) = \hat{D}(\alpha, \beta, \gamma)\hat{j}_3\hat{D}^{-1}(\alpha, \beta, \gamma) \tag{36}$$

where the unitary rotation operator $\hat{D}$ depends on the Euler angles $\alpha,\ \beta,\ \gamma$. The role of the observable $\hat{Z}$ is now played by the spin projection $\hat{J}_3$, while the rotation–transformation parameters are the Euler angles $\sigma_1 = \alpha$, $\sigma_2 = \beta$ and $\sigma_3 = \gamma$. The transformation $\hat{U}(\sigma)$ is given by the matrix representation of the rotation group, i.e. by the Wigner D-function [28].

The marginals are

$$w(s, \alpha, \beta, \gamma) = \langle\langle js|\hat{\rho}|js\rangle\rangle \tag{37}$$

where the rotated spin states become

$$|js\rangle\rangle = \sum_{m=-j}^{j} D_{s\,m}^{(j)\,*}(\alpha, \beta, \gamma)|jm\rangle. \tag{38}$$

Here, the matrix elements $D_{m'\,m}^{(j)}(\alpha, \beta, \gamma)$ (Wigner D-functions) are the matrix elements of the rotation-group representation [28]

$$D_{m'm}^{(j)}(\alpha, \beta, \gamma) = \mathrm{e}^{\mathrm{i}m'\gamma}\, d_{m'm}^{(j)}(\beta)\, \mathrm{e}^{\mathrm{i}m\alpha} \tag{39}$$

where

$$d_{m'm}^{(j)}(\beta) = \left[\frac{(j+m')!(j-m')!}{(j+m)!(j-m)!}\right]^{1/2} \left(\cos\frac{\beta}{2}\right)^{m'+m} \left(\sin\frac{\beta}{2}\right)^{m'-m} P_{j-m'}^{(m'-m,m'+m)}(\cos\beta) \tag{40}$$

with $P_n^{(a,b)}(x)$ being the Jacobi polynomials [28].

Moreover, the transformed spin projector will be

$$\hat{\Pi}_s(\alpha, \beta, \gamma) = \hat{D}(\alpha, \beta, \gamma)|js\rangle\langle js|\hat{D}^{-1}(\alpha, \beta, \gamma) = |js\rangle\rangle\langle\langle js|. \tag{41}$$

Then, we have

$$w(s, \alpha, \beta, \gamma) = \sum_{m_1=-j}^{j} \sum_{m_2=-j}^{j} D_{s\,m_1}^{(j)}(\alpha, \beta, \gamma)\, \rho_{m_1\,m_2}^{(j)}\, D_{s\,m_2}^{(j)\,*}(\alpha, \beta, \gamma). \tag{42}$$

Since

$$D_{m'm}^{(j)*}(\alpha, \beta, \gamma) = (-1)^{m'-m} D_{-m'-m}^{(j)}(\alpha, \beta, \gamma) \tag{43}$$

the marginal distribution really depends only on two angles, α and β. Hence

$$w(s, \alpha, \beta, \gamma) \to w(s, \alpha, \beta) \tag{44}$$

which satisfies the normalization condition

$$\sum_{s=-j}^{j} w(s, \alpha, \beta) = 1. \tag{45}$$

As an example, for a spin-$\frac{1}{2}$ state with spin projection equal to $\frac{1}{2}$, we have

$$\hat{\rho} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \tag{46}$$

and the marginal distributions are

$$w\big(s = \tfrac{1}{2}, \alpha, \beta\big) = \cos^2 \frac{\beta}{2} \qquad w\big(s = -\tfrac{1}{2}, \alpha, \beta\big) = \sin^2 \frac{\beta}{2}. \tag{47}$$

In [17, 24], in view of the properties of the Wigner D-function and the Clebsch–Gordan coefficients, equation (42) was inverted and the density matrix was expressed in terms of the marginal distribution

$$\rho^{(j)}_{m_1 m_2} = (-1)^{m_2} \sum_{j_3=0}^{2j} \sum_{m_3=-j_3}^{j_3} (2j_3+1)^2 \sum_{s=-j}^{j} \int (-1)^s w(s, \alpha, \beta) \times D^{(j_3)}_{0 m_3}(\alpha, \beta, \gamma) W^{j\ \ j\ \ j_3}_{s\ -s\ 0}\, W^{j\ \ \ j\ \ \ \ j_3}_{m_1\ -m_2\ m_3} \frac{\mathrm{d}\Omega}{8\pi^2} \tag{48}$$

where $m_1, m_2 = -j, -j+1, \ldots, j$ and $W^{j_1\ j_2\ j_3}_{m_1\ m_2\ m_3}$ are the Wigner-3j symbols [28]. The integration is performed over the rotation parameters, i.e.

$$\int \mathrm{d}\Omega = \int_0^{2\pi} \mathrm{d}\alpha \int_0^{\pi} \sin\beta\, \mathrm{d}\beta \int_0^{2\pi} \mathrm{d}\gamma. \tag{49}$$

Equation (48) can be presented in an invariant operator form [24]. We systematically introduce the following notation, first for the function on the unit sphere

$$\Phi^{(j_3)}_{j\, m_1 m_2}(\alpha, \beta) = (-1)^{m_2} \sum_{m_3=-j_3}^{j_3} D^{(j_3)}_{0 m_3}(\alpha, \beta, \gamma) W^{j\ \ \ j\ \ \ \ j_3}_{m_1\ -m_2\ m_3} \tag{50}$$

and then for the operator on the unit sphere

$$\hat{A}^{(j_3)}_j(\alpha, \beta) = (2j_3+1)^2 \sum_{m_1=-j}^{j} \sum_{m_2=-j}^{j} |jm_1\rangle\, \Phi^{(j_3)}_{j\, m_1 m_2}(\alpha, \beta)\, \langle jm_2|. \tag{51}$$

In order to write the final expression for the density operator, we introduce an operator on the unit sphere, which contains a dependence on the measurable projection of the spin

$$\hat{K}^{(j)}(s, \alpha, \beta) = (-1)^s \sum_{j_3=0}^{2j} W^{j\ \ j\ \ j_3}_{s\ -s\ 0}\, \hat{A}^{(j_3)}_j(\alpha, \beta). \tag{52}$$

Finally, we obtain a compact expression for the density operator

$$\hat{\rho}^{(j)} = \sum_{s=-j}^{j} \int \frac{\mathrm{d}\Omega}{8\pi^2}\, w(s, \alpha, \beta) \hat{K}^{(j)}(s, \alpha, \beta). \tag{53}$$

Formula (53) admits the following interpretation. To determine the spin state for a spin j, one has to measure experimentally the projection s of the spin for each direction specified by the angles α and β, obtaining a distribution function $w(s, \alpha, \beta)$. The sum on the right-hand side of equation (53) for a given point on the unit sphere represents the average operator $\langle \hat{K}^{(j)}(s, \alpha, \beta)\rangle$. Then, the integral over the whole solid angle gives the desired density operator. Finally, we recognize that, for the spin case, the operator (52) plays the role of the operator $\hat{K}(z, \sigma)$ of equation (13), employed in the general scheme of section 2.

6. The general case

We are now able to consider the case of a particle with $N-1$ spatial degrees of freedom plus one spin-$\frac{1}{2}$ degree. In this case, the state vector $|v\rangle$ has the form

$$|\vec{q}, m\rangle = |q_1, \ldots q_{N-1}\rangle \otimes |\tfrac{1}{2}, m\rangle \tag{54}$$

where $\vec{q}$ is the eigenvalue of the position operator $\hat{\vec{q}}$ and the spin projection $m = (-\frac{1}{2}, \frac{1}{2})$ is the eigenvalue of the Pauli matrix $\hat{\sigma}_z$.

The transformation operator $\hat{U}(\sigma)$ used to construct the tomography scheme, for this case, depends on $2(N-1)$ parameters, determining the symplectic transform, and on three Euler angles determining the spin rotation.

The transformation operator $\hat{U}(\sigma)$ of equation (6) becomes the product of operators

$$\hat{U}(\sigma) = \bigotimes_{k=1}^{N-1} \hat{U}(\mu_k, \nu_k) \otimes \hat{U}(\alpha, \beta, \gamma). \tag{55}$$

For the case of spin $\frac{1}{2}$, the representation of the rotation group is given by

$$D(\alpha, \beta, \gamma) = \begin{pmatrix} \mathrm{e}^{\mathrm{i}\alpha/2} \cos(\beta/2)\, \mathrm{e}^{-\mathrm{i}\gamma/2} & -\mathrm{e}^{-\mathrm{i}\alpha/2} \sin(\beta/2)\, \mathrm{e}^{\mathrm{i}\gamma/2} \\ \mathrm{e}^{\mathrm{i}\alpha/2} \sin(\beta/2)\, \mathrm{e}^{-\mathrm{i}\gamma/2} & \mathrm{e}^{-\mathrm{i}\alpha/2} \cos(\beta/2)\, \mathrm{e}^{\mathrm{i}\gamma/2} \end{pmatrix} \tag{56}$$

which determines the operator

$$\hat{U}(\alpha, \beta, \gamma) = \sum_{m_1=-1/2}^{1/2} \sum_{m_2=-1/2}^{1/2} D^{(1/2)}_{m_1 m_2}(\alpha, \beta, \gamma) |\tfrac{1}{2}, m_1\rangle\langle\tfrac{1}{2}, m_2|. \tag{57}$$

The marginal distribution $w(z, \sigma)$ (12) depends on $N-1$ continuous (non-compact) variables $z_1 = x_1, \ldots, z_{N-1} = x_{N-1}$ and one discrete spin projection $z_N = s$, as well as on the parameters μ_k, ν_k and on the Euler angles α, β. The dependence of the marginal distribution on the Euler angle γ disappears, as was shown in the previous section, due to the structure of Wigner D-functions.

In order to obtain an analogue of the Pauli evolution equation for the marginal distribution, we consider the general equation (16) where the operator $\hat{K}(z', \sigma')$ has the form

$$\hat{K}(z', \sigma') = \frac{1}{8\pi^2} \bigotimes_{k=1}^{N-1} \hat{K}(x_k, \mu_k, \nu_k) \otimes \hat{K}^{(1/2)}(s, \alpha, \beta). \tag{58}$$

Here, the operator $\hat{K}(x_k, \mu_k, \nu_k)$ has the form of equation (31) with $\epsilon = 1$, and the operator $\hat{K}^{(1/2)}(s, \alpha, \beta)$ is given by formula (52) with $j = \frac{1}{2}$. Moreover, we have to introduce the marginal distribution $w(\vec{x}, \vec{\mu}, \vec{\nu}, s, \alpha, \beta, t)$ describing a state of a spin-$\frac{1}{2}$ particle, which depends on the continuous variables $\vec{x}$, discrete spin projection s, symplectic reference frame's labels $\vec{\mu}$ and $\vec{\nu}$, and Euler angles α and β. Then, for a given Hamiltonian $\hat{H}$, the general equation (16) takes the form of a Pauli-like equation

$$\begin{aligned} \partial_t\, w(\vec{x}, \vec{\mu}, \vec{\nu}, s, \alpha, \beta, t) = \sum_{s'=-1/2}^{1/2} \int \mathrm{d}\vec{X}'\, \mathrm{d}\vec{\mu}'\, \mathrm{d}\vec{\nu}'\, \mathrm{d}\Omega'\, w(\vec{x}', \vec{\mu}', \vec{\nu}', s', \alpha', \beta', t) \\ \times \Theta(\vec{x}, \vec{\mu}, \vec{\nu}, s, \alpha, \beta; \vec{x}', \vec{\mu}', \vec{\nu}', s', \alpha', \beta') \end{aligned} \tag{59}$$

where

$$\Theta = \frac{\mathrm{i}}{8\pi^2} \langle\langle \vec{x}, s| \left[\bigotimes_{k=1}^{N-1} \hat{K}(x'_k, \mu'_k, \nu'_k) \otimes \hat{K}^{(1/2)}(s', \alpha', \beta'), \hat{H} \right] |\vec{x}, s\rangle\rangle. \tag{60}$$

The structure of the derived Pauli-like equation for probability distributions depends on the particular tomography schemes we have considered. Obviously, it would be useful to find schemes, which give the simplest form for such a dynamical equation; nevertheless, this is a non-trivial problem related to the possibility to find a properly transformed projector (11). The properties of such projectors were investigated in [21] but for different purposes.

6.1. Limit cases

Now we consider two limit cases of the above equation (59).

First of all, we consider the spatial (one-dimensional) case of free motion

$$\hat{H} = \frac{\hat{p}^2}{2}. \tag{61}$$

The spin part does not contribute since $\hat{H}$ does not contain the spin operators, that is,

$$\begin{aligned}\int \frac{\mathrm{d}\Omega'}{8\pi^2}\, w(s', \alpha', \beta', -)\langle\langle s|\hat{K}^{(j)}(s', \alpha', \beta')|s\rangle\rangle &= \sum_{m_1,m_2=-j}^{j} D^{(j)}_{s\,m_1}(\alpha, \beta, \gamma) D^{(j)*}_{s\,m_2}(\alpha, \beta, \gamma)\\ &\quad\times \sum_{j_3=0}^{2j} \sum_{m_3=-j_3}^{j_3} \sum_{s'=-j}^{j} (-1)^{m_2-s'}(2j_3+1)^2 W^{j\ \ j\ \ j_3}_{s'\ -s'\ 0}\, W^{j\ \ \ j\ \ \ j_3}_{m_1\ -m_2\ m_3}\\ &\quad\times \int \frac{\mathrm{d}\Omega'}{8\pi^2}\, w(s', \alpha', \beta', -) D^{(j_3)}_{0\,m_3}(\alpha', \beta', \gamma') = w(s, \alpha, \beta, -)\end{aligned} \tag{62}$$

where $-$ indicates the other possible variables. Then, for what concerns the spatial part, it is important to calculate the commutator between the kernel and the Hamiltonian, given by

$$\left[\mathrm{e}^{\mathrm{i}\mu'\hat{q}}\mathrm{e}^{\mathrm{i}\nu'\hat{p}}, \hat{p}^2\right] = \mathrm{e}^{\mathrm{i}\mu'\hat{q}}\mathrm{e}^{\mathrm{i}\nu'\hat{p}}(-2\mu'\hat{p} - \mu'^2). \tag{63}$$

Now, one can write

$$\begin{aligned}\partial_t\, w(x, \mu, \nu, t) &= \frac{\mathrm{i}}{4\pi}\int \mathrm{d}x'\,\mathrm{d}\mu'\,\mathrm{d}\nu'\, w(X', \mu', \nu', t)\,\mathrm{e}^{-\mathrm{i}X'+\mathrm{i}\mu'\nu'/2}\\ &\quad\times \int \mathrm{d}q\, \langle\langle x|\mathrm{e}^{\mathrm{i}\mu'\hat{q}}\mathrm{e}^{\mathrm{i}\nu'\hat{p}}|q\rangle\langle q|(-2\mu'\hat{p} - \mu'^2)|x\rangle\rangle.\end{aligned} \tag{64}$$

By using the explicit form for the wavefunctions $\langle q|x\rangle\rangle$ (22) together with the homogeneous property (29), it is possible to reduce the above equation to a very simple form

$$\partial_t\, w = \mu\, \partial_\nu\, w \tag{65}$$

which was derived in a different way in [13].

As the second case, we study the dynamics of spin-$\frac{1}{2}$ degree only. The Hamiltonian we are interested to consider is

$$\hat{H} = \begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix}. \tag{66}$$

Of course, the spatial degree is not affected, so its variables can be disregarded; this also results from the fact that

$$\frac{1}{2\pi}\int \mathrm{d}x'\,\mathrm{d}\mu'\,\mathrm{d}\nu'\, w(x', \mu', \nu', -)\, \langle\langle x|\mathrm{e}^{\mathrm{i}\mu'\hat{q}}\mathrm{e}^{\mathrm{i}\nu'\hat{p}}|x\rangle\rangle\, \mathrm{e}^{-\mathrm{i}x'+\mathrm{i}\mu'\nu'/2} = w(x, \mu, \nu, -). \tag{67}$$

In this case, the relation between the transformed spin state and the non-transformed one is given by

$$|s\rangle\rangle = \hat{D}^{(1/2)*}_{s\,1/2}(\alpha,\beta)\,|\tfrac{1}{2}\rangle + \hat{D}^{(1/2)*}_{s\,-1/2}(\alpha,\beta)\,|-\tfrac{1}{2}\rangle. \tag{68}$$

Again, the central task is to calculate the commutator between the kernel and the Hamiltonian. It is easy to see that

$$\begin{aligned}\langle\langle s|\big[\hat{K}^{(1/2)}(s',\alpha',\beta'),\hat{H}\big]|s\rangle\rangle &= (-1)^{-s'}\sum_{j_3=0}^{1} W^{\frac{1}{2}\ \frac{1}{2}\ j_3}_{s'\ -s'\ 0}(2j_3+1)^2\\ &\times\sum_{m_1\neq m_2,\,-1/2}^{1/2}(-1)^{m_2}\sum_{m_3=-j_3}^{j_3} D^{(j_3)}_{0\,m_3}(\alpha',\beta',\gamma')\,W^{\frac{1}{2}\ \frac{1}{2}\ j_3}_{m_1\ -m_2\ m_3}\\ &\times(-1)^{(1/2)-m_2}(a-c)D^{(1/2)}_{s\,m_1}(\alpha,\beta,\gamma)D^{(1/2)*}_{s\,m_2}(\alpha,\beta,\gamma).\end{aligned} \tag{69}$$

In view of the properties of the Wigner-$3j$ symbols, we can see that the terms with $j_3 = 0, 1$ and $m_3 = 0$ do not contribute. Thus, we obtain

$$\begin{aligned}\partial_t\, w(\tfrac{1}{2},\alpha,\beta,t) &= \int\frac{\mathrm{d}\Omega'}{8\pi^2}\big[w(\tfrac{1}{2},\alpha',\beta',t) - w(-\tfrac{1}{2},\alpha',\beta',t)\big]\\ &\times\tfrac{3}{2}(a-c)\sin\beta'\sin\beta\sin(\alpha-\alpha')\end{aligned} \tag{70}$$

and, by using the normalization condition, equation (70) can be rewritten as

$$\partial_t\, w(s,\alpha,\beta,t) = 3(a-c)\sin\beta\int\frac{\mathrm{d}\Omega'}{8\pi^2}\, w(s,\alpha',\beta',t)\sin\beta'\sin(\alpha-\alpha') \tag{71}$$

which is similar to the relationship derived in [17] (the difference is due to the degeneracy of the spin-$\frac{1}{2}$ system). It should be noted that the argument s is the same on both sides of equation (71); this is consistent with the fact that $\hat{H}$ in equation (66) does not mix states with different s. On the other hand, it can be easily checked that the sum over s on the right-hand side of equation (71) causes the integral to be equal to zero; this is consistent with the fact that on the left-hand side of (71) we obtain the time derivative of a constant. Also, if $a = c$, the right-hand side of equation (71) is equal to zero, since the Hamiltonian (66) is proportional to the identity and does not produce any evolution.

7. Examples

In the previous section, we discussed the probability of the joint measurement of the spin and spatial variables. Therefore, here we would like to consider some examples involving both variables.

At first, we consider a system with the following Hamiltonian:

$$\hat{H} = \tfrac{1}{2}\big(\hat{p}^2 + \hat{q}^2\big) + \big(|\tfrac{1}{2}\rangle\langle\tfrac{1}{2}| - |-\tfrac{1}{2}\rangle\langle-\tfrac{1}{2}|\big). \tag{72}$$

It could describe, for example, one vibrational degree of a trapped electron plus its spin [29]. The measurability of marginals in this system was investigated in [30]. Here, as a straightforward extension of the arguments of section (6.1), we obtain

$$\begin{aligned}\partial_t\, w(x,\mu,\nu,s,\alpha,\beta) &= (\mu\partial_\nu - \nu\partial_\mu)w(x,\mu,\nu,s,\alpha,\beta)\\ &+6\sin\beta\int\frac{\mathrm{d}\Omega'}{8\pi^2}\, w(x,\mu,\nu,s,\alpha',\beta',t)\sin\beta'\sin(\alpha-\alpha').\end{aligned} \tag{73}$$

Let us now consider an initial entangled state like

$$\Psi(0) = \frac{1}{\sqrt{2}} \left(|0\rangle \otimes |-\tfrac{1}{2}\rangle + |1\rangle \otimes |\tfrac{1}{2}\rangle \right) \tag{74}$$

where $|n\rangle$ represents the number eigenstate of a harmonic oscillator. The following marginal corresponds to the wavefunction given by equation (74):

$$w(x, \mu, \nu, s, \alpha, \beta, t = 0) = \tfrac{1}{2} \left[w_{00\downarrow\downarrow} + w_{11\uparrow\uparrow} + w_{01\downarrow\uparrow} + w_{10\uparrow\downarrow} \right] \tag{75}$$

where

$$\begin{aligned}
w_{00\downarrow\downarrow} &= \frac{1}{\sqrt{\pi(\mu^2+\nu^2)}} \exp\left[-\frac{x^2}{\mu^2+\nu^2} \right] D^{(1/2)*}_{s\,-1/2}(\alpha, \beta, \gamma) D^{(1/2)}_{s\,-1/2}(\alpha, \beta, \gamma) \\
w_{11\uparrow\uparrow} &= \frac{2x^2}{\sqrt{\pi(\mu^2+\nu^2)^3}} \exp\left[-\frac{x^2}{\mu^2+\nu^2} \right] D^{(1/2)*}_{s\,1/2}(\alpha, \beta, \gamma) D^{(1/2)}_{s\,1/2}(\alpha, \beta, \gamma) \\
w_{01\downarrow\uparrow} &= \frac{\mathrm{i}\sqrt{2}x(\nu - \mathrm{i}\mu)}{\sqrt{\pi(\mu^2+\nu^2)^3}} \exp\left[-\frac{x^2}{\mu^2+\nu^2} \right] D^{(1/2)*}_{s\,-1/2}(\alpha, \beta, \gamma) D^{(1/2)}_{s\,1/2}(\alpha, \beta, \gamma) \\
w_{10\uparrow\downarrow} &= w^*_{01\downarrow\uparrow}.
\end{aligned}$$

Then, the solution of the Pauli equation (73) results

$$w(x, \mu, \nu, s, \alpha, \beta, t) = \tfrac{1}{2} \left[w_{00\downarrow\downarrow} + w_{11\uparrow\uparrow} + w_{01\downarrow\uparrow}\mathrm{e}^{3\mathrm{i}t} + w_{10\uparrow\downarrow}\mathrm{e}^{-3\mathrm{i}t} \right]. \tag{76}$$

As the second example, we consider the case of Landau levels [31], i.e. a charged particle moving in the classical magnetic field $\vec{B}$, which is time-independent and axial symmetric. The particle's movement along the axis is free. The Hamiltonian of the transverse motion reads

$$\hat{H} = \tfrac{1}{2} \left[\left(\hat{p}_1 - \hat{A}_1 \right)^2 + \left(\hat{p}_2 - \hat{A}_2 \right)^2 \right] \qquad \hat{\vec{A}} = \left[\vec{B} \times \frac{\hat{\vec{r}}}{2} \right] \tag{77}$$

where $\hat{\vec{r}} = (\hat{q}_1, \hat{q}_2)$ is the radius vector of the particle's centre, $\hat{p}_1$ and $\hat{p}_2$ are the particle's momentum components in the transverse plane. Having $\vec{B}$ along the third axis and choosing $|\vec{B}| = 2$, we obtain

$$\hat{H} = \tfrac{1}{2}\left(\hat{p}_1^2 + \hat{p}_2^2 + \hat{q}_1^2 + \hat{q}_2^2 \right) + \left(\hat{p}_1\hat{q}_2 - \hat{p}_2\hat{q}_1 \right) + \left(|\tfrac{1}{2}\rangle\langle\tfrac{1}{2}| - |-\tfrac{1}{2}\rangle\langle-\tfrac{1}{2}| \right). \tag{78}$$

In this case, the kernel Θ of equation (60) is given by

$$\begin{aligned}
\Theta = \frac{\mathrm{i}}{4\pi^2} \int \frac{\mathrm{d}q_1}{\nu_1} \int \frac{\mathrm{d}q_2}{\nu_2} \exp\left\{ \mathrm{i} \sum_{l=1}^{2} \left[\frac{\mu_l'\nu_l'}{2 - x_l'} + \mu_l(q_l - \nu_l') + \frac{x_l\nu_l' + \mu_l\nu_l'^2 - \mu_l\nu_l' q_l}{\nu_l} \right] \right\} \\
\times \left\{ \sum_{l=1}^{2} \frac{\mu_l'\mu_l q_l}{\nu_l} - \frac{\mu_l' x_l}{\nu_l + \nu_l' q_l} - \frac{\mu_l'^2}{2} - \frac{\nu_l'^2}{2} - \frac{(x_2 - \mu_2 q_2)\nu_1'}{\nu_2} \right. \\
\left. + (\mu_2' q_1 - \mu_2'\nu_1') + \frac{(x_1 - \mu_1 q_1)\nu_2'}{\nu_1} - (\mu_1' q_2 - \mu_1'\nu_2') \right\} \delta_{s,s'}\,\delta(\Omega - \Omega') \\
+ \frac{6}{8\pi^2} \sin\beta \sin\beta' \sin(\alpha - \alpha')\,\delta(\vec{x} - \vec{x}')\,\delta(\vec{\mu} - \vec{\mu}')\,\delta(\vec{\nu} - \vec{\nu}').
\end{aligned} \tag{79}$$

As a non-trivial example, we also consider here an initial state, which is the entangled superposition

$$\Psi(0) = \frac{1}{\sqrt{2}} \left[|0\,0\rangle \otimes |-\tfrac{1}{2}\rangle + |1\,0\rangle \otimes |\tfrac{1}{2}\rangle \right]. \tag{80}$$

It leads to a non-factorizable marginal

$$w(\vec{x},\vec{\mu},\vec{\nu},s,\alpha,\beta,t=0)=\tfrac{1}{2}\left[w_{0000\downarrow\downarrow}+w_{1010\uparrow\uparrow}+w_{0010\downarrow\uparrow}+w_{1000\uparrow\downarrow}\right] \tag{81}$$

where

$$w_{n_1\,n_2\,n'_1\,n'_2,m_1\,m_2}=w_{n_1\,n_2\,n'_1\,n'_2}D^{(1/2)\,*}_{s\,m_1}(\alpha,\beta,\gamma)D^{(1/2)}_{s\,m_2}(\alpha,\beta,\gamma). \tag{82}$$

Here, $m_1=-\frac{1}{2}$ ($m_1=\frac{1}{2}$) replaces the downarrow (uparrow), while the spatial part $w_{n_1\,n_2\,n'_1\,n'_2}$ is calculated explicitly in the appendix.

Equation (59) with the kernel (79) appears extremely cumbersome; however, its solution can be obtained in a straightforward way. In fact, given the initial condition (81), one can try to obtain a solution of the form

$$w(\vec{x},\vec{\mu},\vec{\nu},s,\alpha,\beta,t)=\tfrac{1}{2}\left[w_{0000\downarrow\downarrow}\,f_1(t)+w_{1010\uparrow\uparrow}\,f_2(t)+w_{0010\downarrow\uparrow}\,f_3(t)+w_{1000\uparrow\downarrow}\,f_4(t)\right] \tag{83}$$

where f_j ($j=1,\ldots,4$) are functions of time determined by the condition $f_j(t=0)=1$. Then, the trial function (83) inserted into the Pauli equation yields, after some algebra, a simple system of ordinary differential equations

$$\partial_t f_1(t)=0 \qquad \partial_t f_2(t)=0 \qquad \partial_t f_3(t)=3\mathrm{i}f_3(t) \qquad \partial_t f_4(t)=-3\mathrm{i}f_4(t). \tag{84}$$

Thus, the desired solution results

$$w(\vec{x},\vec{\mu},\vec{\nu},s,\alpha,\beta,t)=\tfrac{1}{2}\left[w_{0000\downarrow\downarrow}+w_{1010\uparrow\uparrow}+w_{0010\downarrow\uparrow}\mathrm{e}^{3\mathrm{i}t}+w_{1000\uparrow\downarrow}\mathrm{e}^{-3\mathrm{i}t}\right]. \tag{85}$$

8. Conclusion

We conclude that it is possible to obtain an evolution equation for the tomographic probabilities (marginal distributions) of an arbitrary tomography scheme. The main result of our paper is the analogue of the Pauli equation for the spin-$\frac{1}{2}$ particle.

The explicit expression for the marginal distribution for a trapped particle, as well as for Landau levels, has been studied. The distributions obey the analogue of the Pauli equation.

The examples considered demonstrate that the usual problems of conventional quantum mechanics can be cast into a form, in which only positive probabilities are used to describe quantum states and their evolution. A possible disadvantage of the approach proposed is a complicated evolution equation (59) but, perhaps, this is the price one ought to pay for the possibility of describing quantum objects in terms of classical probabilities.

The classical space has symmetry properties related to its geometry [32]. To extend our approach to relativistic quantum mechanics, one should take into account the geometry of the space–time described by the Poincaré group.

Our argument can constitute a step further from the Bohr position [33] concerning the inapplicability of classical modes of the description in the quantum domain. In fact, while we believe that quantum mechanics is not classical physics, we retain (some) classical concepts still applicable against counterintuitive notions such as complex wavefunctions.

We also believe that the classical-like formalism developed could be applied to describe quantum mechanical paradoxes, because usually, if there is a paradox in quantum mechanics, there should also be a classical one, perhaps, even worse [32]. These aspects will be investigated in a forthcoming paper, as well as the extension of the presented approach to the relativistic domain [34], in order to find an analogue of the Dirac equation.

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Appendix

The wavefunction of the particle's coherent state in the magnetic field $\vec{B}$ is [35]

$$\Psi_{\alpha,\beta}(q_1,q_2)=\frac{1}{\sqrt{\pi}}\exp\left\{-\frac{q_1^2+q_2^2}{2}-\frac{|\alpha|^2}{2}-\frac{|\beta|^2}{2}-\mathrm{i}\alpha\beta+\beta(q_1+\mathrm{i}q_2)+\mathrm{i}\alpha(q_1-\mathrm{i}q_2)\right\} \tag{A1}$$

where q_1 and q_2 are the particle's coordinates and α and β are complex numbers.

The coherent state (A1) is the superposition of number states [35]

$$\Psi_{\alpha,\beta}(q_1,q_2)=\exp\left(-\frac{|\alpha|^2}{2}-\frac{|\beta|^2}{2}\right)\sum_{n=0}^{\infty}\sum_{n'=0}^{\infty}\frac{\alpha^n\beta^{n'}\Psi_{n\,n'}(q_1,q_2)}{\sqrt{n!n'!}}. \tag{A2}$$

In view of the general relationship between the marginal distribution and the wavefunction [36], we have

$$w(x_1,x_2,\mu_1,\nu_1,\mu_2,\nu_2)=\frac{1}{4\pi^2|\nu_1\nu_2|}\times\left|\iint\exp\left(\frac{\mathrm{i}y_1^2\mu_1}{2\nu_1}-\frac{\mathrm{i}y_1x_1}{\nu_1}+\frac{\mathrm{i}y_2^2\mu_2}{2\nu_2}-\frac{\mathrm{i}y_2x_2}{\nu_2}\right)\Psi_{\alpha\,\beta}(y_1,y_2)\,\mathrm{d}y_1\,\mathrm{d}y_2\right|^2$$

where parameters $\mu_1,\ \nu_1,\ \mu_2,\ \nu_2$, as usual, mark reference frames; then, one obtains for the marginal distribution of the particle's coherent state without spin in the magnetic field the following expression:

$$\begin{aligned}w_{\alpha\,\beta}(x_1,x_2,\mu_1,\nu_1,\mu_2,\nu_2)&=\frac{\exp\left[-|\alpha|^2-|\beta|^2-\mathrm{i}(\alpha\beta-\alpha^*\beta^*)\right]}{\pi\sqrt{\left(\nu_1^2+\mu_1^2\right)\left(\nu_2^2+\mu_2^2\right)}}\\&\quad\times\exp\left\{\frac{(\nu_1+\mathrm{i}\mu_1)(\mathrm{i}\alpha\nu_1+\beta\nu_1-\mathrm{i}x_1)^2+(\nu_1-\mathrm{i}\mu_1)\left(-\mathrm{i}\alpha^*\nu_1\beta^*\nu_1+\mathrm{i}x_1\right)^2}{2\nu_1\left(\mu_1^2+\nu_1^2\right)}\right.\\&\quad\left.+\frac{(\nu_2+\mathrm{i}\mu_2)(\alpha\nu_2+\mathrm{i}\beta\nu_2-\mathrm{i}x_2)^2+(\nu_2-\mathrm{i}\mu_2)\left(\alpha^*\nu_2-\mathrm{i}\beta^*\nu_2+\mathrm{i}x_2\right)^2}{2\nu_2\left(\mu_2^2+\nu_2^2\right)}\right\}.\end{aligned} \tag{A3}$$

Multiplying (A3) by $\exp(|\alpha|^2+|\beta|^2)$ and expanding the expression obtained into power series, we arrive at

$$w_{\alpha\,\beta}(x_1,x_2,\mu_1,\nu_1,\mu_2,\nu_2)\mathrm{e}^{|\alpha|^2}\,\mathrm{e}^{|\beta|^2}=\sum_{n_1,n_2,n_1',n_2'=0}^{\infty}\frac{\alpha^{n_1}(\alpha^*)^{n_2}\beta^{n_1'}(\beta^*)^{n_2'}w_{n_1\,n_2\,n_1'\,n_2'}}{\sqrt{n_1!\,n_2!\,n_1'!\,n_2'!}}. \tag{A4}$$

Taking into account the property of the generating function for multivariate Hermite polynomials [37], namely

$$\exp\left\{-\tfrac{1}{2}\vec{u}M\vec{u}+\vec{u}M\vec{\zeta}\right\}=\sum_{n_1,n_2,n_1',n_2'=0}^{\infty}\frac{\alpha^{n_1}(\alpha^*)^{n_2}\beta^{n_1'}(\beta^*)^{n_2'}}{n_1!\,n_2!\,n_1'!\,n_2'!}\,H^{\{M\}}_{n_1\,n_2\,n_1'\,n_2'}(\vec{\zeta}) \tag{A5}$$

with the vector $\vec{u}=(\alpha,\ \alpha^*,\ \beta,\ \beta^*)$, and comparing (A4) with (A5), we obtain

$$\begin{aligned} & w_{n_1\,n_2\,n_1'\,n_2'}(x_1,x_2,\mu_1,\nu_1,\mu_2,\nu_2)\\ &\quad=\frac{1}{\pi\sqrt{\left(\nu_1^2+\mu_1^2\right)\left(\nu_2^2+\mu_2^2\right)}}\exp\left(-\frac{x_1^2}{\mu_1^2+\nu_1^2}-\frac{x_2^2}{\mu_2^2+\nu_2^2}\right)\frac{H^{\{M\}}_{n_1\,n_2\,n_1'\,n_2'}(\vec{\zeta})}{\sqrt{n_1!\,n_2!\,n_1'!\,n_2'!}} \end{aligned}$$

where the 4×4-matrix M reads

$$M=\begin{pmatrix} M^{(1)} & M^{(2)}\\ M^{(4)} & M^{(3)}\end{pmatrix}.$$

The 2×2-matrices $M^{(r)}$ are given by

$$M^{(r)}_{k,l}=\sum_{j=1}^{2}\frac{\nu_j}{\nu_j+\mathrm{i}(-1)^l\mu_j}(-1)^{j+(r+1)/2}\,\delta_{k+l,\,\mathrm{even}}\qquad r=1,3$$

$$M^{(r)}_{k,l}=\sum_{j=1}^{2}\mathrm{i}\left[\frac{\nu_j}{\nu_j+\mathrm{i}(-1)^l\mu_j}(-1)^{l+(j-1)(r/2-1)}+(-1)^{l-1}\right]\delta_{k+l,\,\mathrm{even}}\qquad r=2,4.$$

The argument of the multivariate Hermite polynomials $\vec{\zeta}=\left(\zeta_1,\ \zeta_1^*,\ \zeta_2,\ \zeta_2^*\right)$ is expressed in terms of the parameters as follows:

$$\zeta_1=\frac{\mathrm{i}x_1}{\sqrt{\mu_1^2+\nu_1^2}}\exp\left(\mathrm{i}\tan^{-1}\frac{\mu_2}{\nu_2}\right)-\frac{x_2}{\sqrt{\mu_2^2+\nu_2^2}}\exp\left(\mathrm{i}\tan^{-1}\frac{\mu_1}{\nu_1}\right)$$

$$\zeta_2=\frac{\mathrm{i}x_2}{\sqrt{\mu_1^2+\nu_1^2}}\exp\left(\mathrm{i}\tan^{-1}\frac{\mu_1}{\nu_1}\right)-\frac{x_1}{\sqrt{\mu_2^2+\nu_2^2}}\exp\left(\mathrm{i}\tan^{-1}\frac{\mu_2}{\nu_2}\right).$$

Taking $n_1=n_2$ and $n_1'=n_2'$, we obtain the marginal distribution for the Landau-level states $|nn'\rangle$

$$\begin{aligned} & w_{n\,n'}(x_1,x_2,\mu_1,\nu_1,\mu_2,\nu_2)\equiv w_{n\,n\,n'\,n'}(x_1,x_2,\mu_1,\nu_1,\mu_2,\nu_2)\\ &\quad=\frac{1}{\pi\sqrt{\left(\nu_1^2+\mu_1^2\right)\left(\nu_2^2+\mu_2^2\right)n!\,n'!}}\exp\left(-\frac{x_1^2}{\mu_1^2+\nu_1^2}-\frac{x_2^2}{\mu_2^2+\nu_2^2}\right)H^{\{M\}}_{n\,n\,n'\,n'}(\vec{\zeta}) \end{aligned}$$

where n is the main quantum number and $n'-n=l$ is the angular-momentum quantum number.

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